Recitation 8

October 15, 2015

Problems

Problem 1. Since $\det(A-4I) = \det \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = 0$, $\lambda_1 = 4$ is an eigenvalue. Since $\det A = 0$, $\lambda_2 = 0$ is another eigenvalue.

Problem 2. The determinant is $(-1)^{3+1} \cdot (-1) \cdot \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -(-1-2\cdot 2) = 5$.

Problem 3. As above, we have $\det A = (-1)^{2+1} \cdot (-1) \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} + (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = 9 - 8 = 1$.
Using row reduction, we get

$$\begin{bmatrix} 0 & 3 & 1 \\ -1 & 2 & -2 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

So the determinant is

$$\begin{vmatrix} 0 & 0 & -1/2 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1/2 \end{vmatrix} = 1$$

Problem 4. Carefully row reduce. The answer should be 9 (if I myself row reduced this matrix carefully enough).

Problem 5. If A is invertible, then $A \cdot A^{-1} = I$. So $1 = \det I = \det(A \cdot A^{-1}) = \det A \det A^{-1}$. Thus $\det A^{-1} = \frac{1}{\det A}$.

Problem 6. Since $\det(2I) = 8$, we get

$$\det(2A^{-2}B^TA^{-1}B^2A^T) = 8 \cdot \frac{1}{\det(A)^2} \cdot \det(B) \cdot \frac{1}{\det A} \cdot \det(B)^2 \cdot \det(A) = -2$$

Problem 7. The characteristic equation is $\det(A - \lambda I) = 0$, which is $\lambda^2 - 7\lambda + 6 = 0$. Thus $\lambda_1 = 1$ and $\lambda_2 = 6$ are the two eigenvalues.
For $\lambda_1 = 1$, $A - I = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$. Thus eigenvectors are of the form $x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
For $\lambda_2 = 6$ we get $A - 6I = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$. Thus eigenvectors are of the form $x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
Eigenvectors corresponding to distinct eigenvalues are linearly independent, so we can pick a basis of $\mathbb{R}^2$ consisting of eigenvectors. For example, $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for a basis.

Problem 8. The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$. It has unique solution $\lambda = 3$. Since $A - 3I = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$, its null space is spanned by the vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so all eigenvectors are non-zero multiples of v. So in this case we can't find a basis consisting of eigenvectors.
Problem 9. Compute $\det(A - \lambda I)$. We get

$$\begin{vmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} -1 & -\lambda \\ 1 & 1 \end{vmatrix} = -\lambda^3 + \lambda$$

Thus the characteristic equation is $\lambda^3 - \lambda = 0$, and it has solutions $\lambda = 0, \pm 1$.

Problem 10. $\det(A - \lambda I) = \lambda^2 - (a + d)\lambda + (ad - bc)$. So the constant coefficient is just $\det A$, and the coefficient of λ is minus the sum of the diagonal elements.

Problem 11. The constant coefficient of $\det(A - \lambda I)$ is just $\det(A)$. The coefficient of λ^2 will be the sum of the diagonal elements.